International Journal of Scientific & Engineering Research, Volume 5, Issue 9, September-2014 5
ISSN 2229-5518

Frame of a Closed Subspace of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ Generated by Translation of a Function

Bijumon Ramalayathil

Abstract— The concept of MRA in wavelet analaysis, MRA wavelets in the superspace $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and Frame MRA in $L^2(\mathbb{R})$ are now well known. In this paper we discuss frame MRA in $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and frame of a closed subspace of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ generated by translation of a function.

Index Terms— Wavelet, MRA wavelets, Bessel map, Frames in Hilbert Space, Frame MRA, Frame of a Closed Subspace of the Superspace.

---

1 Introduction

IN what follows $L^2(\mathbb{R})$ denotes the Hilbert space of square integrable functions defined on $\mathbb{R}$. The superspace $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is the Hilbert space under the inner product defined by

$$\langle (f_1, f_2), (g_1, g_2)\rangle_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} = \langle f_1, g_1\rangle_{L^2(\mathbb{R})} + \langle f_2, g_2\rangle_{L^2(\mathbb{R})}$$

The applications of wavelet theory and frame theory to signal processing and image processing are now well known. Probably the main reason for the success of the wavelet theory was the introudction of the concept of multiresolution analysis (MRA) [1] which provides the right frame work to construct orthogonal wavelt bases with good localization properties. It was shown in [2] that wavelets in the superspaces cannot be constructed through MRA in the usual sense. However, by modifying the usual dilation and translation operators, MRA wavelets in the superspace $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ have been constructed in [3]. Proceeding in the same line, we describe frame of a closed subspace of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ generated by translation of a function and define Frame MRA in the superspace.

2 Definitions

The Fourier transform on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is defined by

$$(f_1, f_2)^{\wedge} = (\hat{f}_1, \hat{f}_2),$$

where

$$\forall \gamma \in \mathbb{R}, \quad \hat{f}_i(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_i(t) e^{-it\gamma} dt \qquad i = 1, 2.$$

Taking $\omega = \exp(-2\pi i/3)$, $C = \{z_1 = \omega,\ z_2 = \omega^2\}$ is a cycle [4]; i.e., a periodic orbit for the map $z \mapsto z^2$ on the unit circle T. The translation operator T_C and dilation operator U_C on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ are given by

$$T_C(f_1 \oplus f_2) = \omega T f_1 \oplus \omega^2 T f_2, \qquad (1)$$

$$U_C(f_1 \oplus f_2) = U f_2 \oplus U f_1 \qquad (2)$$

where T and U are the translation and dilation operators on $L^2(\mathbb{R})$ defined by:

$$Tf(x) = f(x-1) \text{ and } (Uf)(x) = \sqrt{2} f(2x).$$

With $e_k(t) = e^{ikt}$, for $f \in L^2(\mathbb{R})$ we have

$$\widehat{T^k f} = e_k \hat{f}$$

and

$$\widehat{Uf} = U^{-1}\hat{f}.$$

A countable collection $X = \{x_i \oplus x_i' : i \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\|f_1 \oplus f_2\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \le \sum_{i\in\mathbb{Z}} \left|\langle f_1 \oplus f_2, x_i \oplus x_i'\rangle\right|^2_{l^2(\mathbb{Z})}$$

$$\le B\|f_1 \oplus f_2\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})},$$

for every $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. If $A = B$, the frame is a tight frame.

Let $X = \{x_i \oplus x_i' : i \in \mathbb{Z}\}$ be a countable system in the separable Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. If the map

$$\mathcal{L} : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to l^2(\mathbb{Z})$$

$$f_1 \oplus f_2 \mapsto \{\langle f_1 \oplus f_2, x_i \oplus x_i'\rangle : i \in \mathbb{Z}\}$$

is well defined in the sense that it takes values in $l^2(\mathbb{Z})$, then $\mathcal{L}$ is the **Bessel map** associated with X. Whenever the Bessel map $\mathcal{L}$ exists, $\mathcal{L}$ is bounded by the uniform boundedness principle [5]. The adjoint of $\mathcal{L}$ is given by

$$\mathcal{L}^* : l^2(\mathbb{Z}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

$$c \mapsto \sum_{i\in\mathbb{Z}} c_i x_i \oplus x'.$$

3 Results

The following result is a special case of the proposition in [6].

Proposition 1.

Let $X = \{x_i \oplus x_i' : i \in \mathbb{Z}\} \subseteq L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ be a countable system with a well-defined Bessel map $\mathcal{L} : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to l^2(\mathbb{Z})$.

• *Bijumon Ramalayathil is currently Assistant Professor in Post Graduate Department of Mathematics, Mahatma Gandhi College, Iritty, Kerala, India, PH-07560967926. E-mail:bijumon.iritty@gmail.com*

ISSN 2229-5518

Assume $\overline{\text{span}}\{x_i \oplus x_i' : i \in \mathbb{Z}\} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Then for every $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$,

$$A\|f_1 \oplus f_2\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \le \sum_{i\in\mathbb{Z}} \left|\langle f_1 \oplus f_2, x_i \oplus x_i'\rangle\right|^2_{l^2(\mathbb{Z})} \le B\|f_1 \oplus f_2\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})}$$

if and only if for every $c \in \left(N(L^*)\right)^{\perp}$,

$$A\|c\|^2_{l^2(\mathbb{Z})} \le \|\mathcal{L}^*(c)\|^2_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})} \le B\|c\|^2_{l^2(\mathbb{Z})}.$$

i.e., X is a frame if and only if $\mathcal{L}^*$ is bounded and invertible.

Proposition 2.
Let $X = \left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and define

$$\Phi_1(\gamma) = \sum_{k\in\mathbb{Z}} \left|\hat{\phi}_1(\gamma + 2\pi k)\right|^2 \quad \text{and} \quad \Phi_2(\gamma) = \sum_{k\in\mathbb{Z}} \left|\hat{\phi}_2(\gamma + 2\pi k)\right|^2.$$

Assume that the Bessel map $\mathcal{L}$ associated with X exists. If $\Phi_1 \le A$ and $\Phi_2 \le B$ a.e., then $\|\mathcal{L}\| \le (A+B)^{1/2}$. Conversely, $\|\mathcal{L}\| \le C^{1/2}$ implies $\Phi_1 \le C$ and $\Phi_2 \le C$ a.e.

Proof Let c be a finitely generated sequence. Then

$$\|\mathcal{L}^*(c)\|^2 = \|\mathcal{L}^*(c)\|^2 = \left\|\sum_{k\in\mathbb{Z}} c_k T_C^k \phi_1 \oplus \phi_2\right\|^2 = \int_{\mathbf{T}} \left|\sum_{k\in\mathbb{Z}} c_k \omega^k e_k\right|^2 \Phi_1 + \int_{\mathbf{T}} \left|\sum_{k\in\mathbb{Z}} c_k \omega^{2k} e_k\right|^2 \Phi_2.$$

Note that $\sum_{k\in\mathbb{Z}} c_k \omega^k e_k$ is the Fourier transform of the sequence $(c_k\omega^k) \in l^2(\mathbb{Z})$ and $\sum_{k\in\mathbb{Z}} c_k \omega^{2k} e_k$ is the Fourier transform of the sequence $(c_k\omega^{2k}) \in l^2(\mathbb{Z})$. Hence, by the Parseval-Plancherel theorem for $\mathbf{T}$ [7], we have

$$\|c_k\omega^k\|^2_{l^2(\mathbb{Z})} = \int_{\mathbf{T}} \left|\sum_{k\in\mathbb{Z}} c_k \omega^k e_k\right|^2$$

and

$$\|c_k\omega^{2k}\|^2_{l^2(\mathbb{Z})} = \int_{\mathbf{T}} \left|\sum_{k\in\mathbb{Z}} c_k \omega^{2k} e_k\right|^2$$

and both equal to $\|(c_k)\|^2_{l^2(\mathbb{Z})}$.

Thus, if $\Phi_1 \le A$ and $\Phi_2 \le B$ a.e. on $\mathbf{T}$, then

$$\|\mathcal{L}^*\| < A + B.$$

Since $\|\mathcal{L}\| = \|\mathcal{L}^*\|$, it follows that $\|\mathcal{L}\| \le (A+B)^{1/2}$.

For the converse, consider for $\delta > 0$ the set $\Gamma = [\Phi_1 \ge C + \frac{\delta}{2}] \cup [\Phi_2 \ge C + \frac{\delta}{2}]$. Now, for any measurable set $\Gamma \subseteq \mathbf{T}$, there exists a sequence $\{p_n\}$ of trigonometric polynomials with $\|p_n\|^2_{L^2(\mathbf{T})} \le |\Gamma|$ such that $\{p_n\}$ converges to χ_Γ (the characteristic function of the set Γ) except on a set of arbitrarily small measure. Thus, if the measure $|\Gamma|$ of Γ was strictly greater than 0, there would be a finitely supported sequence c with $\|c\|^2_{l^2(\mathbb{Z})} \le |\Gamma|$ such that

$$\|\mathcal{L}^*(c)\|^2 \ge |\Gamma|(C + \tfrac{\delta}{2}).$$

Hence

$$\|\mathcal{L}^*\|^2 \ge C.$$

Theorem 3.
Let $X = \left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and assume $\Phi_1(\gamma) = \sum_{k\in\mathbb{Z}} \left|\hat{\phi}_1(\gamma + 2\pi k)\right|^2$ and $\Phi_2(\gamma) = \sum_{k\in\mathbb{Z}} \left|\hat{\phi}_2(\gamma + 2\pi k)\right|^2 \in L^\infty(\mathbf{T})$.
Then X has a well defined Bessel map $\mathcal{L}$. Further, taking

$$A_1 = \inf\left\{a : \left|[\Phi_1 \le a] \cap [\Phi_1 > 0]\right| > 0\right\},$$
$$A_2 = \inf\left\{a : \left|[\Phi_2 \le a] \cap [\Phi_2 > 0]\right| > 0\right\},$$
$$\text{esssup}\Phi_1 = B_1 < \infty,$$

and

$$\text{esssup}\Phi_2 = B_2 < \infty,$$

X is a frame for $V_0 = \overline{\text{span}}\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ with lower frame bound greater than or equal to $A_1 + A_2 > 0$ and upper frame bound less than or equal to $B_1 + B_2 < \infty$. [This frame is the frame generated by $\phi_1 \oplus \phi_2$.]

Proof
Φ_1 and Φ_2 are essentially bounded implies that there are $C_1 > 0$ and $C_2 > 0$ such that $|\Phi_1(x)| < C_1$ and $|\Phi_2(x)| < C_2$ a.e. Taking

$$\frac{C}{2} = \max\{C_1, C_2\}$$

we obtain

$$\Phi_1 \le \frac{C}{2} \quad \text{and} \quad \Phi_2 \le \frac{C}{2} \quad \text{a.e.}$$

Hence by Proposition 2,

$$\|\mathcal{L}\| < C^{1/2}.$$

Hence $\mathcal{L}$ takes values in $l^2(\mathbb{Z})$ and hence $\mathcal{L}$ is well defined. By the definition of V_0, $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is complete in V_0. For $c \in l^2(\mathbb{Z})$

$$\|\mathcal{L}^*(c)\|^2 = \int_{\mathbf{T}} \left|\sum_{k\in\mathbb{Z}} c_k \omega^k e_k\right|^2 \Phi_1 + \int_{\mathbf{T}} \left|\sum_{k\in\mathbb{Z}} c_k \omega^{2k} e_k\right|^2 \Phi_2.$$

Now for the values A_1, A_2, B_1, B_2 chosen above,

$$(A_1 + A_2)\|c\|^2_{l^2(\mathbb{Z})} \le \|\mathcal{L}^*(c)\|^2 \le (B_1 + B_2)\|c\|^2_{l^2(\mathbb{Z})}.$$

Consequently, X is a frame with lower frame bound greater than or equal to $A_1 + A_2$ and upper frame bound less than or equal to $B_1 + B_2$.

Corollary 4. For X, V_0, A_1, A_2, B_1 and B_2 as in Theorem 3, if $A_1 + A_2 = B_1 + B_2$, then X is a tight frame for V_0 with bound

$A_1 + A_2$.

Proposition 5. Suppose $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is a frame for its closed span V_0. Then

$$f_1 \oplus f_2 \in V_0 \Rightarrow \hat{f}_1 = F_1 \hat{\phi}_1 \text{ and } \hat{f}_2 = F_2 \hat{\phi}_2,$$

for some $F_1, F_2 \in L^2(\mathbf{T})$ depending on $f_1 \oplus f_2 \in L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.
Conversely, if $\hat{f}_1 = F_1 \hat{\phi}_1$ and $\hat{f}_2 = F_2 \hat{\phi}_2$ with $F_1 = (c_k \omega^k)\hat{}$ and $F_2 = (c_k \omega^{2k})\hat{}$ for some sequence $(c_k) \in l^2(\mathbb{Z})$, then $f_1 \oplus f_2 \in V_0$.

Proof

Since $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for its closed span V_0, $f_1 \oplus f_2 \in V_0$ implies

$$f_1 \oplus f_2 = \sum_{k \in \mathbb{Z}} c_k T_C^k \phi_1 \oplus \phi_2$$

for some sequence $c = (c_k) \in l^2(\mathbb{Z})$.

Fourier transform of this equation gives

$$\hat{f}_1 = F_1 \hat{\phi}_1 \quad \text{and} \quad \hat{f}_2 = F_2 \hat{\phi}_2,$$

where $F_1 = \sum_{k \in \mathbb{Z}} c_k \omega^k e_k$ and $F_2 = \sum_{k \in \mathbb{Z}} c_k \omega^{2k} e_k$. The fact that F_1 and $F_2 \in L^2(\mathbf{T})$ follows from Parseval's theorem [7], noting that F_1 is the Fourier transform of the sequence $(c_k \omega^k)$ and F_2 that of $(c_k \omega^{2k})$.

Conversely, if

$$\hat{f}_1 = \sum_{k \in \mathbb{Z}} c_k \omega^k e_k \hat{\phi}_1 \quad \text{and} \quad \hat{f}_2 = \sum_{k \in \mathbb{Z}} c_k \omega^{2k} e_k \hat{\phi}_2,$$

then

$$\hat{f}_1 = \sum_{k \in \mathbb{Z}} c_k \omega^k \left(T^k \phi_1\right)^{\wedge} \quad \text{and} \quad \hat{f}_2 = \sum_{k \in \mathbb{Z}} c_k \omega^{2k} \left(T^k \phi_2\right)^{\wedge}.$$

Hence

$$f_1 = \sum_{k \in \mathbb{Z}} c_k \omega^k T^k \phi_1 \quad \text{and} \quad f_2 = \sum_{k \in \mathbb{Z}} c_k \omega^{2k} T^k \phi_2$$

and so

$$f_1 \oplus f_2 = \sum_{k \in \mathbb{Z}} c_k T_C^k \phi_1 \oplus \phi_2 \in V_0.$$

Definitiion 1. A **frame MRA (FMRA)** $\left\{V_j, \phi_1 \oplus \phi_2\right\}$ of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ is an increasing sequence of closed subspaces $V_j \subset L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and an element $\phi_1 \oplus \phi_2 \in V_0$ for which the following hold:

(1) $\bigcup_j V_j = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ and $\bigcap_j V_j = \{(0,0)\}$,

(2) $f_1 \oplus f_2 \in V_j \Leftrightarrow U_C f_1 \oplus f_2 \in V_{j+1} \quad \forall j \in \mathbb{Z}$,

(3) $f_1 \oplus f_2 \in V_0 \Leftrightarrow T_C^k f_1 \oplus f_2 \in V_0 \quad \forall k \in \mathbb{Z}$,

(4) $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for the subspace V_0.

Proposition 6. Let $\left\{V_j, \phi_1 \oplus \phi_2\right\}$ be an FMRA. If $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_0, then the system $\left\{U_C^j T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_j with the same frame bounds.

Proof

$\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_0 implies that there exist A and B (both greater than zero) such that for all $f_1 \oplus f_2 \in V_0$,

$$A\left\| f_1 \oplus f_2 \right\|^2 \le \sum_{k \in \mathbb{Z}} \left| \left\langle f_1 \oplus f_2, T_C^k \phi_1 \oplus \phi_2 \right\rangle \right|^2 \le B \left\| f_1 \oplus f_2 \right\|^2.$$

Also, for $j \in \mathbb{Z}$,

$$\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subseteq V_0 \Leftrightarrow \left\{U_C^j T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\} \subseteq V_j.$$

Also,

$$\left(U_C^j\right)^* = U_C^{-j} \text{ for } j \in \mathbb{Z}.$$

Hence for $f_1 \oplus f_2 \in V_j$,

$$\sum_{k \in \mathbb{Z}} \left| \left\langle f_1 \oplus f_2, U_C^j T_C^k \phi_1 \oplus \phi_2 \right\rangle \right|^2 = \sum_{k \in \mathbb{Z}} \left| \left\langle U_C^{-j} f_1 \oplus f_2, T_C^k \phi_1 \oplus \phi_2 \right\rangle \right|^2. \quad (1)$$

Since $U_C^{-j} f_1 \oplus f_2 \in V_0$ and $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_0, we have

$$A\left\| U_C^{-j} f_1 \oplus f_2 \right\|^2 \le \sum_{k \in \mathbb{Z}} \left| \left\langle U_C^{-j} f_1 \oplus f_2, T_C^k \phi_1 \oplus \phi_2 \right\rangle \right|^2 \le B \left\| U_C^{-j} f_1 \oplus f_2 \right\|^2.$$

But,

$$\left\| U_C^{-j} f_1 \oplus f_2 \right\|^2 = \left\| f_1 \oplus f_2 \right\|^2.$$

Hence, using (1), we obtain

$$A\left\| f_1 \oplus f_2 \right\|^2 \le \sum_{k \in \mathbb{Z}} \left| \left\langle f_1 \oplus f_2, U_C^j T_C^k \phi_1 \oplus \phi_2 \right\rangle \right|^2 \le B \left\| f_1 \oplus f_2 \right\|^2.$$

Thus $\left\{U_C^j T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_j with the same frame bounds of the frame $\left\{T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ of V_0.

Proposition 7. Suppose $\left\{V_j, \phi_1 \oplus \phi_2\right\}$ be an FMRA of $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. Then there are 2π-periodic functions H_0' and H_0'' belong to $L^2(\mathbf{T})$ such that

$$\hat{\phi}_1(2\cdot) = H_0' \hat{\phi}_2$$

and

$$\hat{\phi}_2(2\cdot) = H_0'' \hat{\phi}_1.$$

Also

$$\Phi_1 = \left| H_0'(\tfrac{\cdot}{2}) \right|^2 \Phi_2(\tfrac{\cdot}{2}) + \left| H_0'(\tfrac{\cdot}{2} + \pi) \right|^2 \Phi_2(\tfrac{\cdot}{2} + \pi)$$

and

$$\Phi_2 = \left| H_0''(\tfrac{\cdot}{2}) \right|^2 \Phi_1(\tfrac{\cdot}{2}) + \left| H_0''(\tfrac{\cdot}{2} + \pi) \right|^2 \Phi_1(\tfrac{\cdot}{2} + \pi).$$

Proof V_1 is closed and invariant under translations. So $V_0 \subseteq V_1$ if and only if $\phi_1 \oplus \phi_2 \in V_1$. As $\left\{U_C T_C^k \phi_1 \oplus \phi_2 : k \in \mathbb{Z}\right\}$ is a frame for V_1, we have

$$\phi_1 \oplus \phi_2 = \sum_{k \in \mathbb{Z}} c_k U_C T_C^k \phi_1 \oplus \phi_2.$$

Taking the Fourier transforms on both sides, we obtain

$$\hat{\phi}_1 = \sum_{k \in \mathbb{Z}} c_k U^{-1} (\omega^{2k} T^k \phi_2)^{\wedge},$$

and

$$\hat{\phi}_2 = \sum_{k \in \mathbb{Z}} c_k U^{-1} (\omega^{k} T^k \phi_1)^{\wedge},$$

implies

ISSN 2229-5518

$$\hat{\phi}_1(2\cdot) = H_0'\hat{\phi}_2$$

and

$$\hat{\phi}_2(2\cdot) = H_0''\hat{\phi}_1,$$

where $H_0' = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} c_k \omega^{2k} e_k$ and $H_0'' = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} c_k \omega^{k} e_k$ both belong to $L^2(\mathbf{T})$.

Now we periodize the square modulus of

$$\hat{\phi}_1(\cdot) = H_0'\hat{\phi}_2(\tfrac{\cdot}{2}):$$

$$\sum_{k\in\mathbb{Z}}\left|\hat{\phi}_1(\cdot + 2\pi k)\right|^2 = \sum_{k\in\mathbb{Z}}\left|H_0'\left(\tfrac{\cdot+2\pi k}{2}\right)\right|^2\left|\hat{\phi}_2\left(\tfrac{\cdot+2\pi k}{2}\right)\right|^2$$

$$= \left|H_0'\left(\tfrac{\cdot}{2}\right)\right|^2\sum_{k\in\mathbb{Z}}\left|\hat{\phi}_2\left(\tfrac{\cdot}{2} + 2\pi k\right)\right|^2$$

$$+\left|H_0'\left(\tfrac{\cdot}{2} + \pi\right)\right|^2\sum_{k\in\mathbb{Z}}\left|\hat{\phi}_2\left(\tfrac{\cdot}{2} + \pi + 2\pi k\right)\right|^2$$

That is,

$$\Phi_1 = \left|H_0'(\tfrac{\cdot}{2})\right|^2\Phi_2(\tfrac{\cdot}{2}) + \left|H_0'(\tfrac{\cdot}{2}+\pi)\right|^2\Phi_2(\tfrac{\cdot}{2}+\pi) \; .$$

Similarly, we obtain

$$\Phi_2 = \left|H_0''(\tfrac{\cdot}{2})\right|^2\Phi_1(\tfrac{\cdot}{2}) + \left|H_0''(\tfrac{\cdot}{2}+\pi)\right|^2\Phi_1(\tfrac{\cdot}{2}+\pi).$$

4 Conclusion

In this paper the concept of Bessel map has been used to define Frame MRA and for the discussion of frame generated by translation of a function in a closed subspace of the superspace. The study of these concepts are important in the construction of FMRA in the superspaces.

Acknowledgment

The author wish to thank Dr M.S. Balasubramani, Professor (Retd.), Department of Mathematics, University of Calicut, Kerala.

References

1. S. Mallat, "A Theory of Multiresolution Approximations and Wavelet Orthornomral Basis of $L^2(\mathbb{R})$," Tran. Amer. Math. Soc., **315** 69-87 (1989).
2. D. Han and D. Larson, "Frames, bases and group representations," Memoirs of the AMS, Volume 147, Number **697**, (2000).
3. Stefan Bildea and Dorin Ervin Dutkay and Gabriel Picioroaga, "MRA Super-Wavelets", New York Journal of Mathematics, 11:1-19, (2005).
4. D. E. Dutkay, "Positive definite maps, representations and frames", Reviews in Mathematical Physics, **164** (2004), 1-27.
5. Walter Rudin, *"Functional Analysis"* (Second Edition), Tata McGraw-Hill Publishing Company Ltd., New Delhi-8.
6. John J. Benedetto and Oliver M. Treiber, "Wavelet Frames: Multiresolution Analysis and Extension Principle", Birakauhser, *Wavelet Transforms and Time-Frequency Signal Analysis* -ed. Lokenath Debnath Pages Nos.3-36.
7. Jayakumar Ramanathan, "*Methods of Applied Fourier Analysis"*, Birkhauser, Boston.